揺動 Wilberforce 振子の強制振動の基礎解析

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Fundamental Analysis of Swinging Wilberforce Pendulum with External Forcing

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Abstract: In view of various types of pendulums provided at foreign universities and colleges, some representative education materials are reexamined to show fundamental mechanical phenomena and to step up to higher level of PBL (Problem Based Learning) resources. As one of such typical pendulums, Wilberforce pendulum is taken up here, whose longitudinal and torsional modes of oscillation are known to arise from the nature of coil spring and may give rise to near resonance. The pendulum is allowed to swing in a vertical plane to lead the system to three degrees of freedom. Nonlinear free oscillation is simulated at near resonance and near the critical. It is found that initial values of the swinging causes the coupling oscillation among the three oscillation modes and the near resonance occurs in the three modes.

Keywords: Wilberforce Pendulum, Longitudinal and Torsional Mode of Oscillation, Coupling Oscillation, Near Resonance

1 はじめに

Wilberforce 振子[2]に用いられるコイルバネでは，素線をヘリカルに巻いてあるので，伸縮と捩れによる連成振動が起こる．この振子は，連成振動系の代表例の一つとして振子教材 list[3] にも挙げられており，幾つかの実験解説書や研究報告[1],[4]–[13] がある．特に，Hughey[14] は共振 (resonance) について理論的実験的検証を行っており，共振と準共振 (near resonance，あるいは近共振) について考察している．この現象は伸縮と捩れの結合バネ定数 ϵ が小さいときに起こり，$\epsilon \to 0$ の極限が共振点になる．

本研究では，準共振を区別する視点から，振子教材の充実・整備のために，Wilberforce 振子の基礎運動解析?、揺動 Wilberforce 振子の運動[16]、壁面からの励振による揺動 Wilberforce 振子の運動[17] を解明して来た。それらに続き、本報告では、揺動 Wilberforce 振子が外力により強制連成振動する場合を解析し、計算例を示す．先ず，揺動 Wilberforce 振子の自由振動を記述する Lagrange 関数を導き，運動方程式を求める．この運動では系の力学的 energy は保存量である．その線形振動解を解析的に求めると共に数値解析を行い、非線形振動の特性を調べる．

2 揺動 Wilberforce 振子の解析

Fig.1 に示すように，壁面上の原点から水平方向に x 軸、鉛直下方を z 軸とするデカルト座標系 (x, z) を取り、平面極座標系 (r, θ) を併用して，壁面から吊り下げられた自然長 r_0 で伸縮のバネ定数 k_1 の線形バネの下端に取り付けられた質量 m_1 の物体 (錘) の質量中心までの距離を $r_1 \equiv r_1(t)$ (t: 時間) と表し，鉛直軸から測るバネの傾き角を $\theta_1 \equiv \theta_1(t)$ と表す．線形バネの下端に取り付けられた質量 m_1 の物体 (錘) の質量中心までの距離を $r_1 \equiv r_1(t)$ (t: 時間) と表し，鉛直軸から測るバネの傾き角を $\theta_1 \equiv \theta_1(t)$ と表す．線形バネの捩りのバネ定数を δ とし，バネの捩れ角を $\psi_1 \equiv \psi_1(t)$ と表し、バネの伸縮と捩れの結合バネ定数を ϵ とする．錘は、質量 m_1 で捩れの慣性 moment J_1 を持ち、鉛直面内で r 方向に伸縮運動し、θ 方向に揺動 (swing) し、ψ 方向に捩れ振動する．錘の座標 (x_p, z_p) は，支持点 (x_0, z_0) を一定値とし、次のように表される:

$$x_p = x_0 + r_1 \sin(\theta_1), \quad z_p = z_0 + r_1 \cos(\theta_1) \tag{2.1}$$

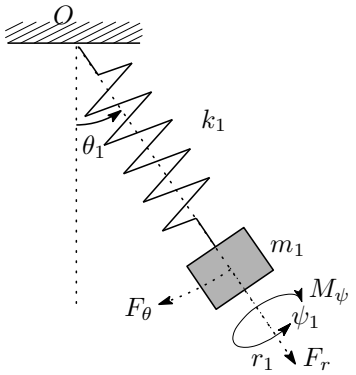

Fig.1 Wilberforce pendulum.

(2.1) 式により、揺動 Wilberforce 振子の運動を記述する Lagrange 関数 $\mathcal{L}_2$ は，次のように表される：

$$\mathcal{L}_2 = \frac{m_1}{2}\left(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2\right) + \frac{J_1}{2}\dot{\psi}_1^2 - \frac{k_1}{2}\left(r_1 - r_0\right)^2 - \frac{\delta}{2}\psi_1^2 - \frac{\epsilon}{2}\left(r_1 - r_0\right)\psi_1 + m_1 g\left[r_1\left(\cos(\theta_1) - 1\right) + z_0\right] \tag{2.2}$$

これにより，Lagrange の運動方程式は，次のように表さ

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[*4] 機械工学科: Department of Mechanical Engineering.
[*5] 電気電子工学科: Department of Electrical & Electronics Engineering.

れる：

$$
\begin{aligned}
& m_1\ddot{r}_1 + k_1(r_1 - r_0) + \frac{\epsilon}{2}\psi_1 - m_1 g(\cos(\theta_1) - 1) \\
& - m_1 r_1 \dot{\theta}_1^2 = -c_r\dot{r}_1 + F_r\cos(\omega t), && (2.3) \\
& m_1 r_1 \ddot{\theta}_1 + m_1 g\sin(\theta_1) + 2m_1\dot{r}_1\dot{\theta}_1 = -c_\theta\dot{\theta}_1 \\
& + F_\theta\cos(\omega t), && (2.4) \\
& J_1\ddot{\psi}_1 + \delta\psi_1 + \frac{\epsilon}{2}(r_1 - r_0) = -c_\psi\dot{\psi}_1 + M_\psi\cos(\omega t) && (2.5)
\end{aligned}
$$

但し、θ_1 方向の運動方程式を求めて r_1 で除して (2.4) 式の左辺としたので注意されたい。

3 揺動 Wilberforce 振子の強制振動の数値解析

3.1 Wilberforce-swing-Fr-july27-20.tex

Wilberforce 振子の規定値を $g = 1$, $m_1 = 1$, $J_1 = 1$, $k_1 = 1$, $\delta = 1$, $\epsilon = 0.1$, $r_0 = 1$, $c_r = 0.1$, $c_\theta = 0.1$,$c_\psi = 0.1$, $F_r = 1$, $F_\theta = 0$, $F_\psi = 0$, $\omega = 1$ とおく．微分方程式系 (2.3), (2.4), (2.5) 式の初期値を $r_1(0) = 1$, $\dot{r}_1(0) = 0$, $\theta_1(0) = 0.1$, $\dot{\theta}_1(0) = 0$, $\psi_1(0) = 0$, $\dot{\psi}_1(0) = 0$ と設定して，(2.3), (2.4), (2.5) 式を時間 $t_s \le t \le t_e$ ($t_s = 2\pi \times 20$, $t_e = 2\pi \times 32$) で数値積分した結果を **Fig.2a-2h** に示す.

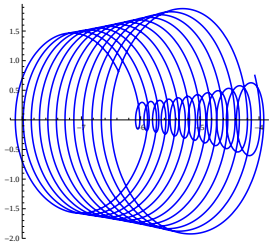

Fig.2a Phase portrait of $(r_1, \dot{r}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).

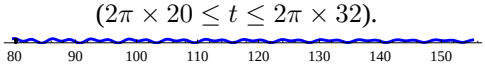

Fig.2b Phase portrait of $(\theta_1, \dot{\theta}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).

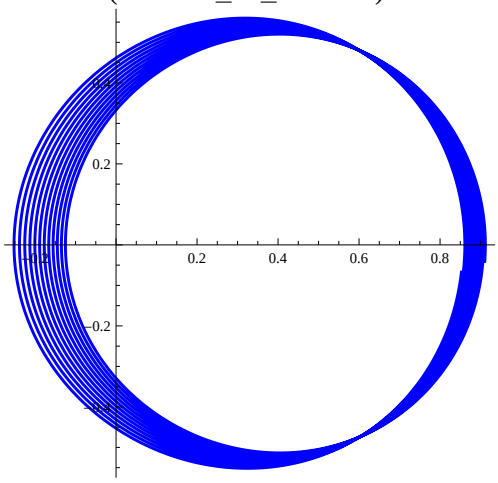

Fig.2c Phase portrait of $(\psi_1, \dot{\psi}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).

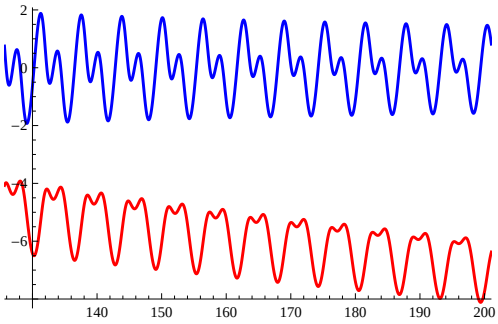

Fig.2d Time sequence of $r_1(t)$ and $\dot{r}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

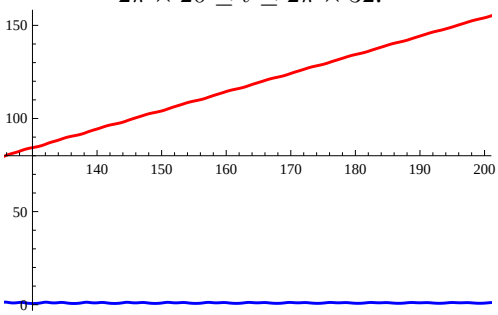

Fig.2e Time sequence of $\theta_1(t)$ and $\dot{\theta}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

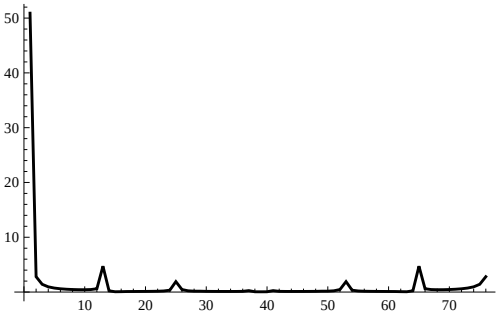

Fig.2f Spectrum of $r_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

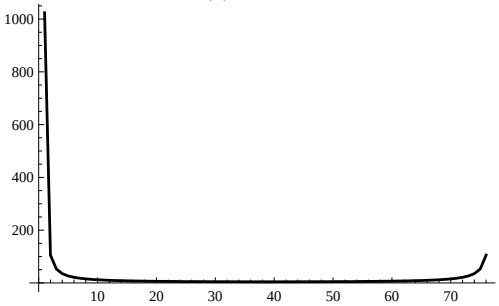

Fig.2g Spectrum of $\theta_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

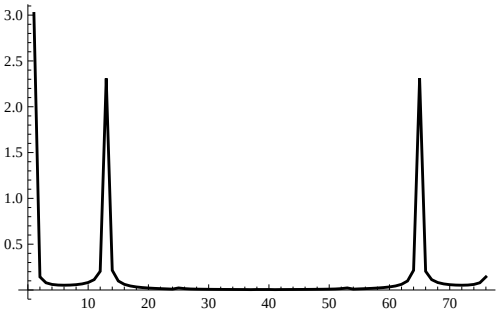

Fig.2h Spectrum of $\psi_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

3.2 Wilberforce-swing-Ft-july27-20.tex

Wilberforce 振子の規定値を $g = 1$, $m_1 = 1$, $J_1 = 1$, $k_1 = 1$, $\delta = 1$, $\epsilon = 0.1$, $r_0 = 1$, $c_r = 0.1$, $c_\theta = 0.1$,$c_\psi = 0.1$, $F_r = 0$, $F_\theta = 1$, $F_\psi = 0$, $\omega = 1$ とおく．微分方程式系 (2.3), (2.4), (2.5) 式の初期値を $r_1(0) = 1$, $\dot{r}_1(0) = 0$, $\theta_1(0) = 0$, $\dot{\theta}_1(0) = 0$, $\psi_1(0) = 0$, $\dot{\psi}_1(0) = 0$ と設定して，(2.3), (2.4), (2.5) 式を時間 $t_s \le t \le t_e$ ($t_s = 2\pi \times 20$, $t_e = 2\pi \times 32$) で数値積分した結果を **Fig.3a-3i** に示す．

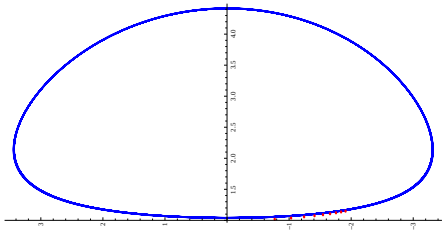

Fig.3a Phase portrait of $(r_1, \dot{r}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).The figure is rotated by 90°.

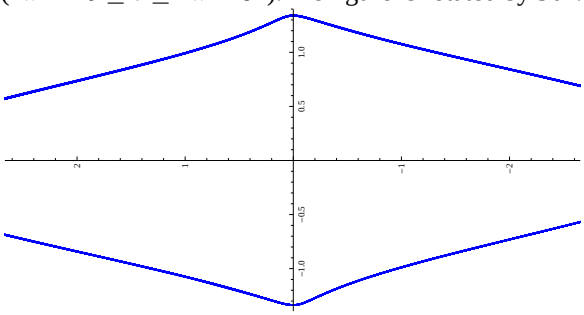

Fig.3b Phase portrait of $(\theta_1, \dot{\theta}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).The figure is rotated by 90°.

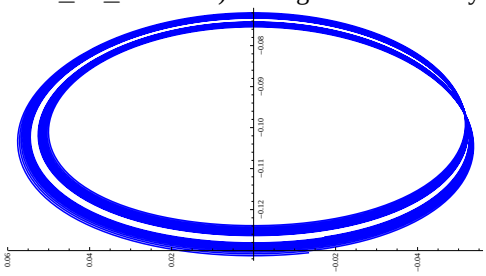

Fig.3c Phase portrait of $(\psi_1, \dot{\psi}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).The figure is rotated by 90°.

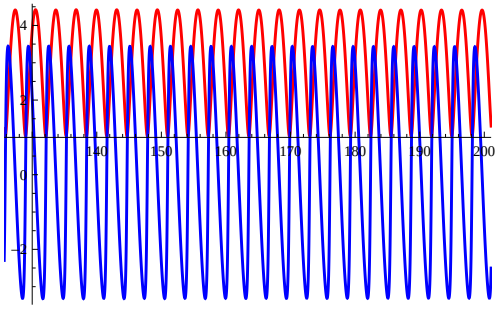

Fig.3d Time sequence of $r_1(t)$ and $\dot{r}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

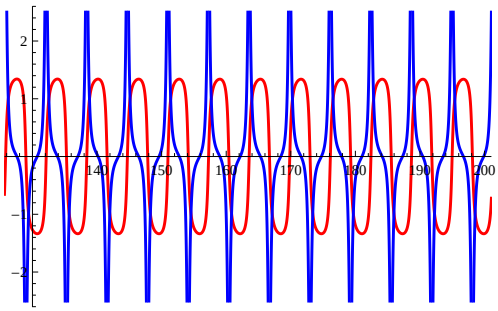

Fig.3e Time sequence of $\theta_1(t)$ and $\dot{\theta}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

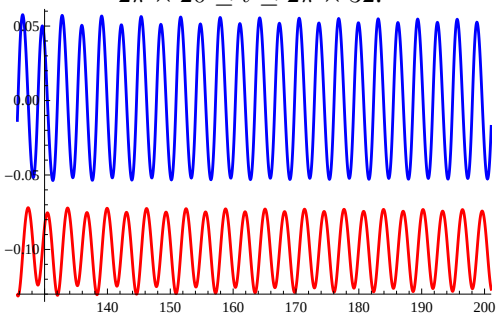

Fig.3f Time sequence of $\psi_1(t)$ and $\dot{\psi}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

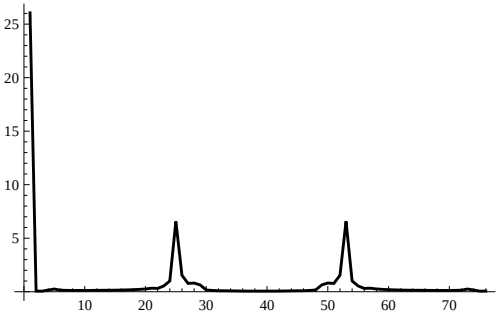

Fig.3g Spectrum of $r_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

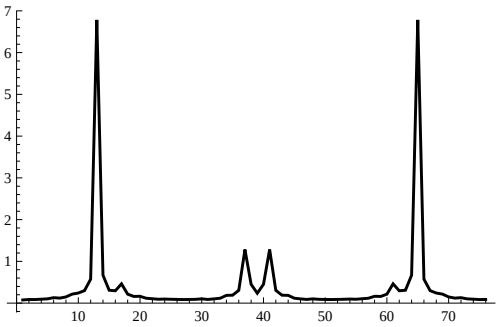

Fig.3h Spectrum of $\theta_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

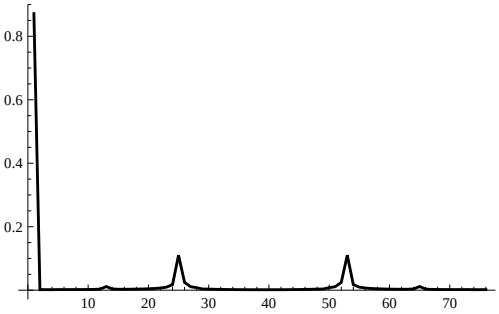

Fig.3i Spectrum of $\psi_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

3.3 Wilberforce-swing-Fp-july27-20.tex

Wilberforce 振子の規定値を $g = 1$, $m_1 = 1$, $J_1 = 1$, $k_1 = 1$, $\delta = 1$, $\epsilon = 0.1$, $r_0 = 1$, $c_r = 0.1$, $c_\theta = 0.1$,$c_\psi = 0.1$, $F_r = 0$, $F_\theta = 0$, $F_\psi = 1$, $\omega = 1$ とおく．微分方程式系 (2.3), (2.4), (2.5) 式の初期値を $r_1(0) = 1$, $\dot{r}_1(0) = 0$, $\theta_1(0) = 0.1$, $\dot{\theta}_1(0) = 0$, $\psi_1(0) = 0$, $\dot{\psi}_1(0) = 0$ と設定して，(2.3), (2.4), (2.5) 式を時間 $t_s \le t \le t_e$ ($t_s = 2\pi \times 20$, $t_e = 2\pi \times 32$) で数値積分した結果を **Fig.4a-4i** に示す．

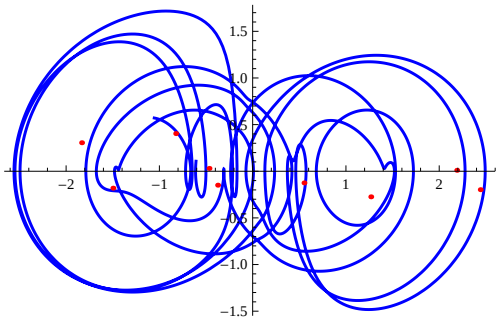

Fig.4a Phase portrait of $(r_1, \dot{r}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).

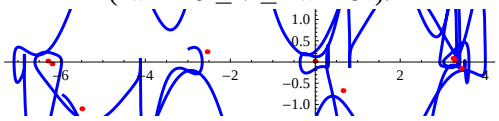

Fig.4b Phase portrait of $(\theta_1, \dot{\theta}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).

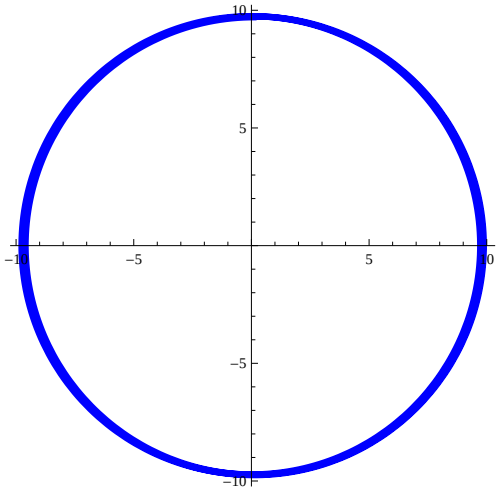

Fig.4c Phase portrait of $(\psi_1, \dot{\psi}_1)$ versus t ($2\pi \times 20 \le t \le 2\pi \times 32$).

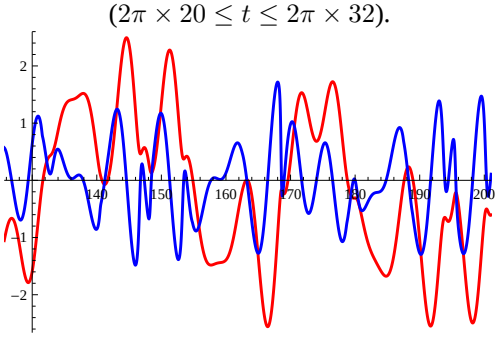

Fig.4d Time sequence of $r_1(t)$ and $\dot{r}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

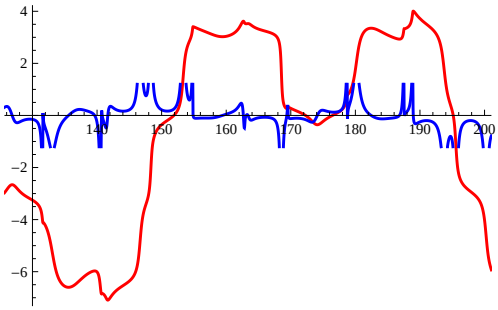

Fig.4e Time sequence of $\theta_1(t)$ and $\dot{\theta}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

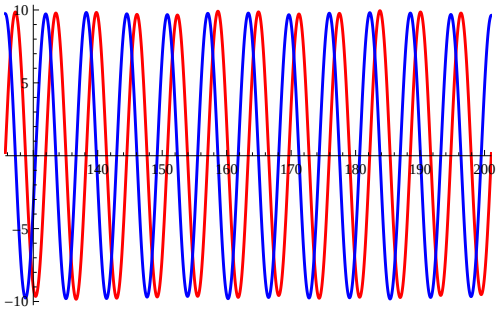

Fig.4f Time sequence of $\psi_1(t)$ and $\dot{\psi}_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

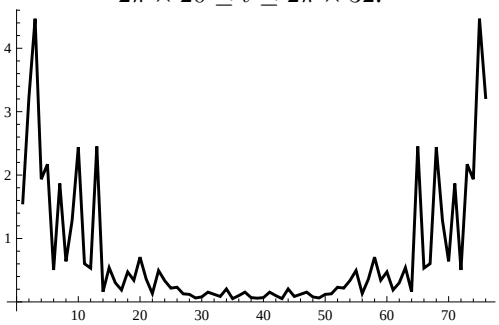

Fig.4g Spectrum of $r_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

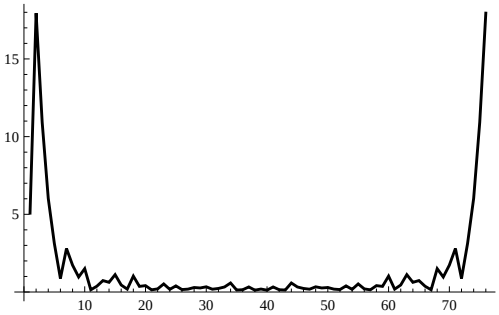

Fig.4h Spectrum of $\theta_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

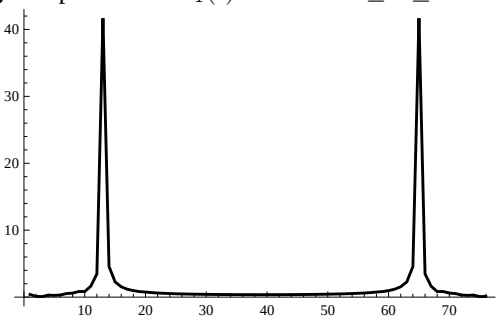

Fig.4i Spectrum of $\psi_1(t)$ in $2\pi \times 20 \le t \le 2\pi \times 32$.

4 おわりに

本報告では，Wilberforce 振子の基礎解析を行い，代表的な場合について数値解析を行った．系の規定値を代表的な値に設定したときのバネの伸縮と捩れの結合バネ定数 ϵ が小さいときに準共振が起こる．系の規定値を設定すると，$\epsilon \sim 0$ は $\omega_1 \sim \omega_2$ の準共振を表すが，$\epsilon \to 0$ では 2 つの振動 mode は独立になる．臨界点 $\epsilon = 2\sqrt{m_1 J_1} = 2$ では $\omega_2 = 0$ となり，系は ω_1 で振動する．$\epsilon > 2$ では系は不安定になる．以上のように，ϵ による振動の型の変化と特徴を詳しく解説した．

先行の実験解説書や研究報告[4]–[13] には Wilberforce 振子の実験的理論的解説があり，諸国の高等教育での教材整備が見られる．しかし，Wilberforce 振子を拡張して，様々な振動問題を解説する取組は以外に少ないようである．本報告に続き，Wilberforce 振子が揺動する場合の解析[16]，支持点の励振[17] や外力と減衰効果[18]，複数の Wilberforce 振子の振動問題の解析[19] に取組む．また，準共振に関して，線形振動 model と非線形振動 model を解析して，別途報告する[24]．なお，準共振の研究報告の調査検討は，引き続き行う．振子の振動する様子の理解には動画教材が効果的であり，インターネット上に供給されている例[1],[6] を挙げることができる．

ここに示した実験データや解析例は学生の自学自習の支援に提供できるのみならず，子供科学教室[1] やものづくり教室，並びに出前授業等で科学技術に関心を呼び起こす展示・教材にも活用できるものと期待される．

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