揺動 Atwood 機械の数値 simulation: 球面振子

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Numerical Simulation of Swinging Atwood's Machine: Spherical Pendulum

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Abstract: Atwood's machine is a representative device in mechanics, but it performs very complex phenomena when one body at its end swings in a vertical plane, which is referred to as swinging Atwood's machine (SAM). This has been extensively studied by Tufillaro and coworkers (N. B. Tufillaro, T. A. Abbott, and D. J. Griffiths, Swinging Atwood's Machine, American Journal of Physics 52 (10) (1984), pp.895-903) to reveal periodic and chaotic motions. We revisited this mechanical device to make PBL (Problem Based Learning) resources and practice problems for numerical computations. As an extension from the original type of swinging Atwood machine, the pendulum may work as a spherical pendulum. This can be solved numerically based on the conservation of angular momentum of rotational mode, and some of the results are reported in the present paper.

Keywords: Swinging Atwood's Machine, Spherical Pendulum, Periodic and Chaotic Motion

1 はじめに

「初等教育から高等教育までの「振子」の教程で，どのような教材を用いて何を教育しているか」を現在の技術者教育の視点から調査研究することを本研究の目的とし，外国の高等教育機関の物理学・工学実験教材も調査して来た[1]．その代表的な教材の一つに「揺動 Atwood 機械 (swinging Atwood's machine, SAM)」がある．揺動 Atwood 機械の特性は，特に Tufillaro[2]–[12] により Hamilton 系の力学問題として精力的に研究されている．Tufillaro[2] は，Atwood の機械の一方の錘 (質量 m) が揺動することにより非常に多彩な現象が現れることを指摘しており、周期解やカオス解を見出している．最近では Yehia[13] の報告があり、傾斜した円錐振子の問題が揺動 Atwood 機械の問題と類似であることを述べている．一方、最近のエレベータの振動問題の解明を契機に、本研究でも，揺動 Atwood 機械の振動基礎解析[14] を行い、揺動 Atwood 機械の支持点からのパラメトリック励振問題を数値解析して来た[15]–[17]。本報告でも，Tufillaro[2] に習い，振子運動の自由度を拡張し、球面振子[18],[19] として SAM の運動方程式の解析と数値計算を行い SAM の特性を解明する．§2 では揺動 Atwood 機械の力学問題を定式化し，§3 では運動方程式の初期値問題を数値解析する．さらに，§4 では設定条件を変えて運動方程式を数値解析する．

2 揺動 Atwood 機械の運動方程式

Fig.1 の左側の質量 M の物体は，左側の滑車から鉛直に吊り下がっており，上下に運動する．右側の質量 m の錘は，右側の滑車からの距離を r，鉛直軸との角度を θ として，鉛直面内で振子運動する．紐の全長を r_0 とすると、物体の鉛直位置は $r_0 - r - c$ である。

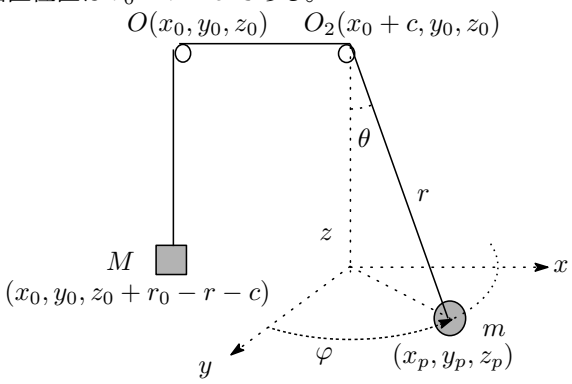

Fig.1 Swinging Atwood's Machine.

球座標 (r, θ, φ) を用い，右側の錘の位置が記述でき，左側の錘の鉛直変位は r で記述できる．それぞれの錘の運動 energy と重力 potential energy を求めると，この系の運動を記述する Lagrange 関数 $\mathcal{L}$ は次のように表わされる:

$$\mathcal{L} = \frac{M}{2}\dot{r}^2 + \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2\right) + Mg(r_0 - r - c) + mrg\cos(\theta) \tag{2.1}$$

質量比 $\mu = M/m$ を導入し，$\mathcal{L}$ を $\mathcal{L}_1$ に書き換える:

$$\mathcal{L}_1 = \frac{\mu}{2}\dot{r}^2 + \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2\right) - gr\left(\mu - \cos(\theta)\right) \tag{2.2}$$

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これにより，Lagrange の運動方程式は次式となる;

$$(1+\mu)\ddot{r} - r\dot{\theta}^2 - r\sin^2(\theta)\dot{\varphi}^2 + g\left(\mu - \cos(\theta)\right) = 0, \quad (2.3)$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin(\theta)\cos(\theta)\dot{\varphi}^2 + gr\sin(\theta) = 0 \quad (2.4)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\sin^2(\theta)\dot{\varphi}\right) = 0$$
$$\rightarrow \alpha_1 = r^2\sin^2(\theta)\dot{\varphi} = \text{constant} \quad (2.5)$$

故に、φ 方向の角運動量 α が保存するので、式中の $\dot{\varphi}$ を消去して、$r(t), \theta(t)$ のみの方程式系を得る:

$$(1+\mu)\ddot{r} - r\dot{\theta}^2 - \frac{\alpha_1^2}{r^3\sin^2(\theta)} + g\left(\mu - \cos(\theta)\right) = 0, \quad (2.6)$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - \frac{\alpha_1^2\cos(\theta)}{r^2\sin^3(\theta)} + gr\sin(\theta) = 0 \quad (2.7)$$

これらは，θ に関して非線形項が含まれている連立非線形微分方程式であり，初期値問題として解くことができる．(2.3), (2.4) 式は，また，自律系の方程式であり，系の力学的 energy が保存するので，有界な領域に解があり，周期解を持つことが示される:

$$\begin{aligned} E_1 &= \frac{\mu}{2}\dot{r}^2 + \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2\right) \\ &\quad + gr\left(\mu - \cos(\theta)\right) \\ &\geq \frac{\mu+1}{2}\dot{r}^2 + \frac{1}{2}r^2\left(\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2\right) \\ &\quad + gr\left(\mu - 1\right) \end{aligned} \quad (2.8)$$

3 数値計算 (1)

揺動 Atwood 機械の規定値を $g = 1$, $\mu = 1.0125$ とおく．微分方程式系 (2.6), (2.7) 式の初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.22$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0$ と設定して，(2.6), (2.7) 式を時間 $0 \leq t \leq t_e$ ($t_e = 2\pi \times 32$) で数値積分した結果を **Fig.2a-2d** に示す.

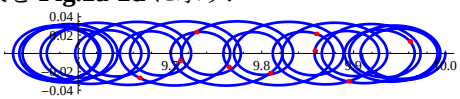

Fig.2a Phase portrait of $(r, \dot{r})$ in $0 \leq t \leq 2\pi \times 32$.

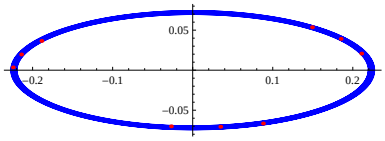

Fig.2b Phase portrait of $(\theta, \dot{\theta})$ in $0 \leq t \leq 2\pi \times 32$.

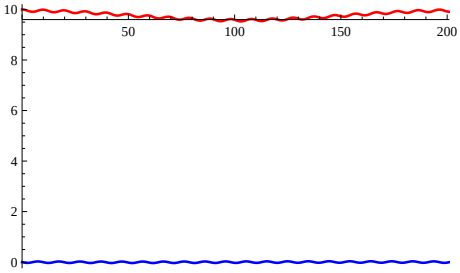

Fig.2c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \leq t \leq 2\pi \times 32$.

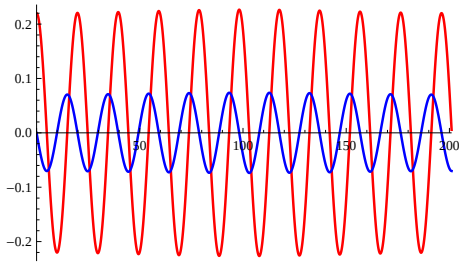

Fig.2d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \leq t \leq 2\pi \times 32$.

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0.001$ と設定して，(2.6), (2.7) 式を時間 $0 \leq t \leq t_e$ ($t_e = 2\pi \times 32$) で数値積分した結果を **Fig.3a-3d** に示す.

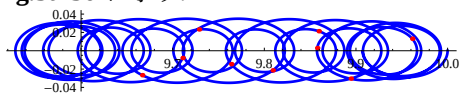

Fig.3a Phase portrait of $(r, \dot{r})$ in $0 \leq t \leq 2\pi \times 32$.

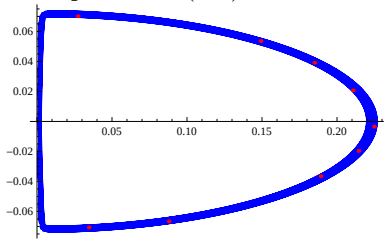

Fig.3b Phase portrait of $(\theta, \dot{\theta})$ in $0 \leq t \leq 2\pi \times 32$.

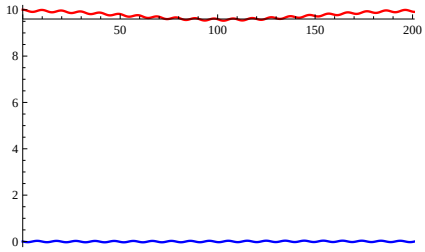

Fig.3c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \leq t \leq 2\pi \times 32$.

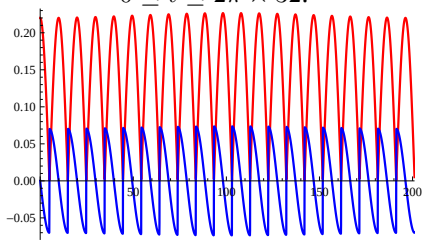

Fig.3d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \leq t \leq 2\pi \times 32$.

系の規定値は同様とし、初期値を $\dot{\varphi}(0) = 0.01$ に換えた場合、数値計算結果は **Fig.3a-3d** と定性的に同じである。

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.22$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0.1$ と設定して，(2.6), (2.7) 式を時間 $0 \leq t \leq t_e$ ($t_e = 2\pi \times 32$) で数値積分した結果を **Fig.4a-4d** に示す.

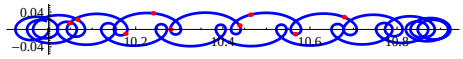

Fig.4a Phase portrait of $(r, \dot{r})$ in $0 \leq t \leq 2\pi \times 32$.

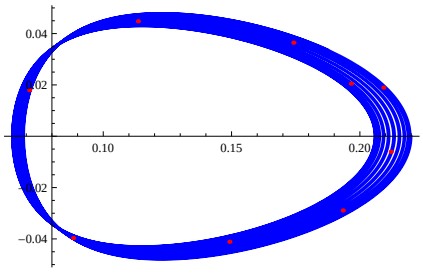

Fig.4b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 32$.

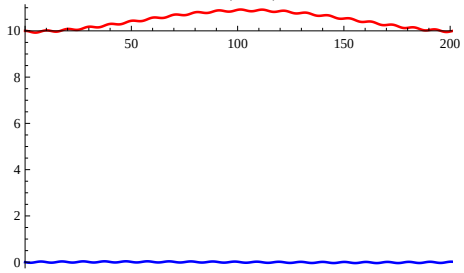

Fig.4c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

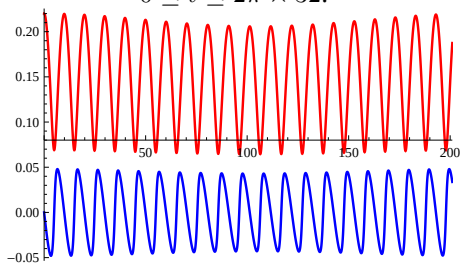

Fig.4d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.22$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 1$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 32$) で数値積分した結果を **Fig.5a-5d** に示す.

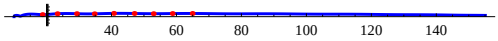

Fig.5a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 32$.

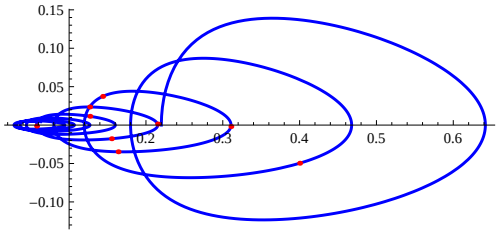

Fig.5b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 32$.

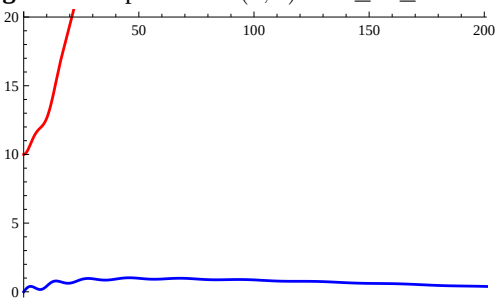

Fig.5c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

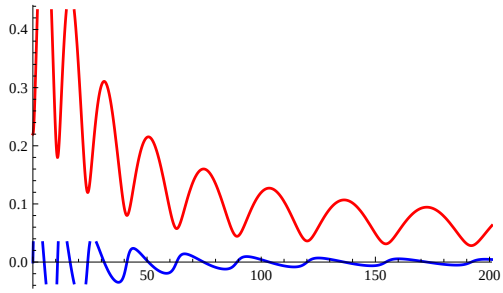

Fig.5d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

4 数値計算 (2)

揺動 Atwood 機械の規定値を $g = 1$, $\mu = 2$ とおく．微分方程式系 (2.6), (2.7) 式の初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.6a-6e** に示す.

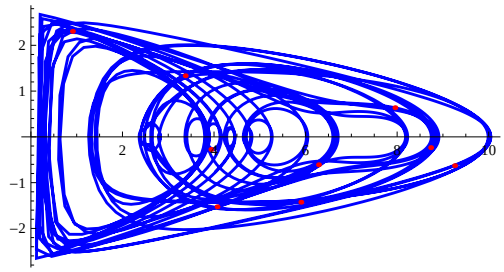

Fig.6a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 62$.

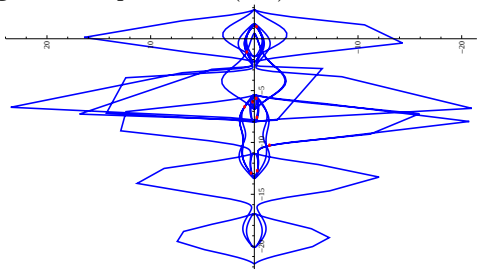

Fig.6b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 62$.

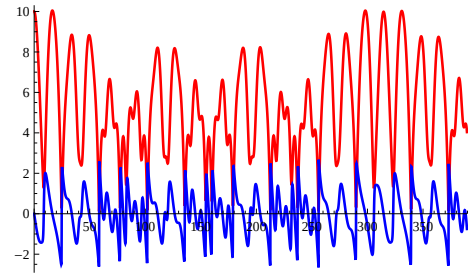

Fig.6c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

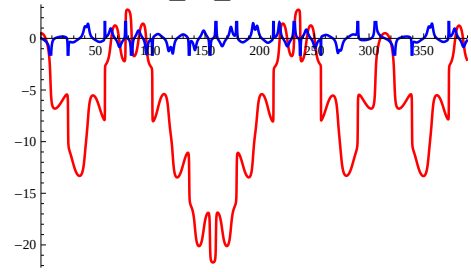

Fig.6d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

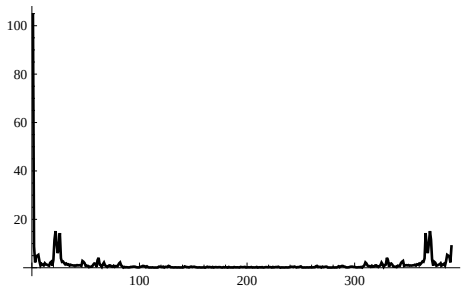

Fig.6e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 62$.

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.6$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0.001$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.7a-7e** に示す.

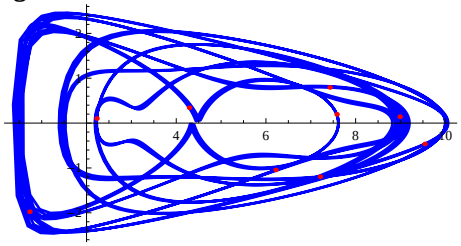

Fig.7a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 62$.

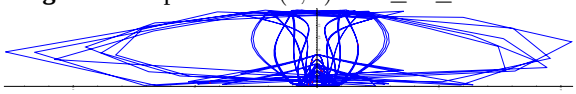

Fig.7b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 62$.The figure is rotated by 90°.

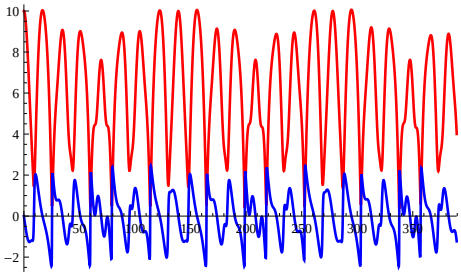

Fig.7c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

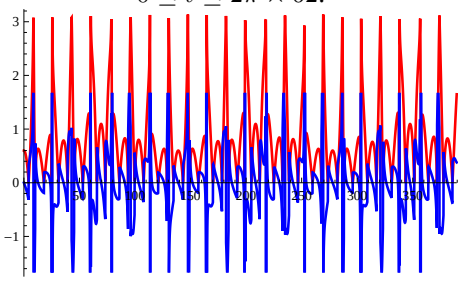

Fig.7d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

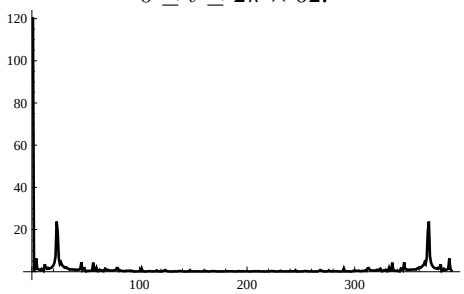

Fig.7e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 62$.

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.7$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0.01$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.8a-8e** に示す

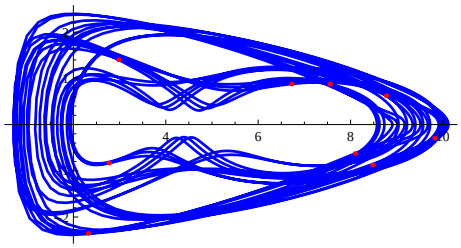

Fig.8a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 62$.

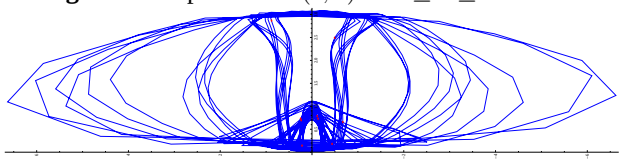

Fig.8b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 62$.The figure is rotated by 90°.

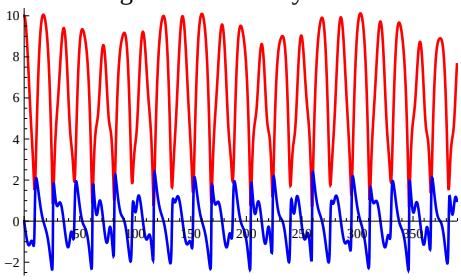

Fig.8c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

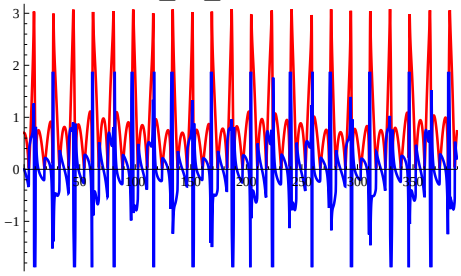

Fig.8d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

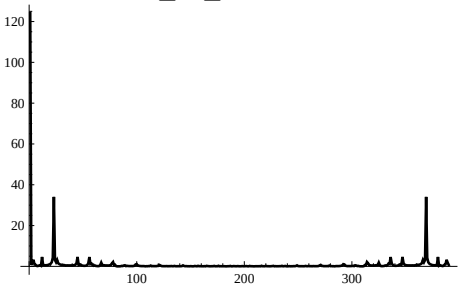

Fig.8e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 62$.

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0.1$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.9a-9e** に示す.

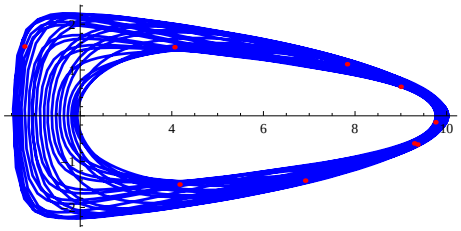

Fig.9a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 62$.

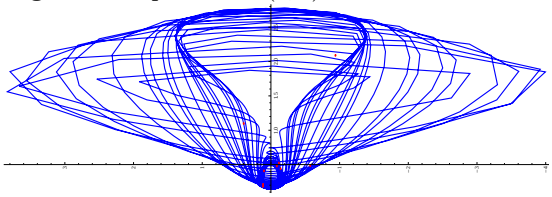

Fig.9b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 62$.The figure is rotated by 90°.

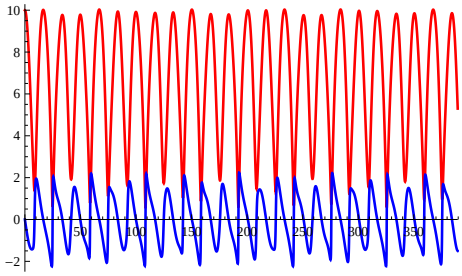

Fig.9c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

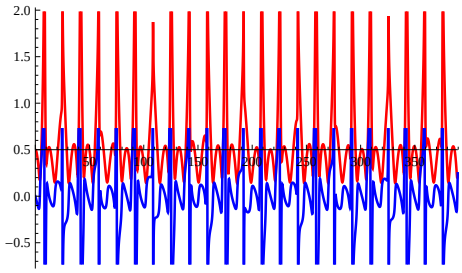

Fig.9d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

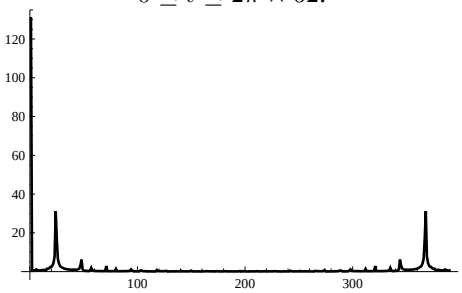

Fig.9e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 62$.

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 1$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.10a-10e** に示す.

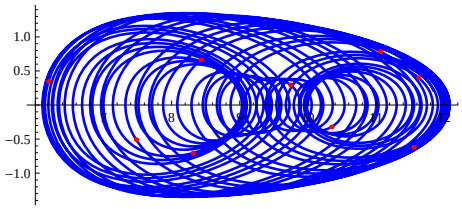

Fig.10a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 62$.

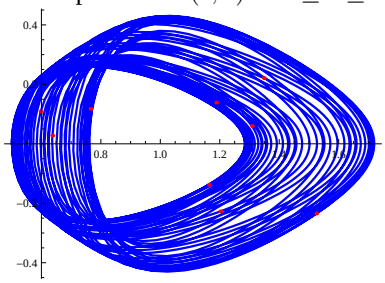

Fig.10b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 62$.

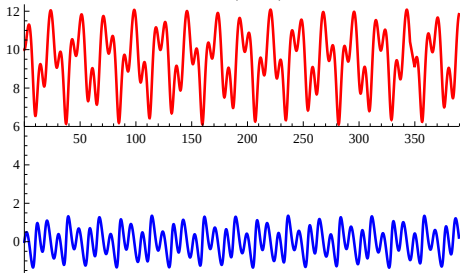

Fig.10c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

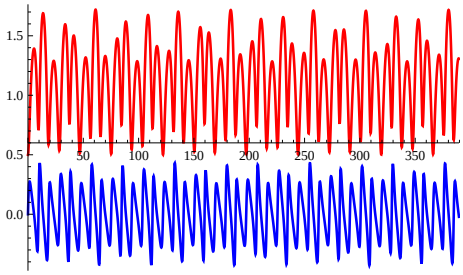

Fig.10d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 62$.

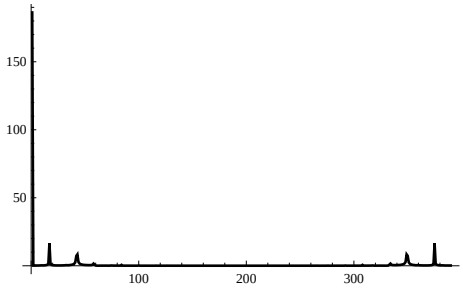

Fig.10e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 62$.

系の規定値を $g = 1$, $\mu = 3$ とおく．微分方程式系 (2.6), (2.7) 式の初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0.1$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.11a-11d** に示す.

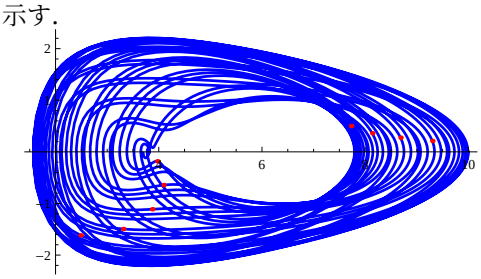

Fig.11a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 32$.

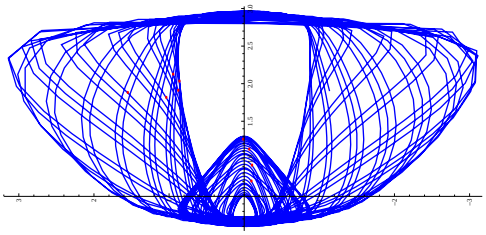

Fig.11b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 32$.The figure is rotated by 90°.

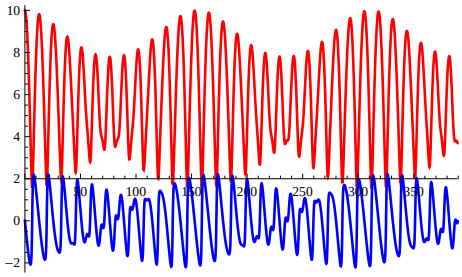

Fig.11c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

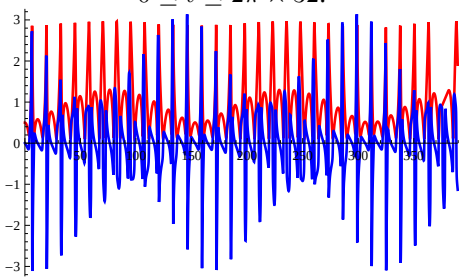

Fig.11d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

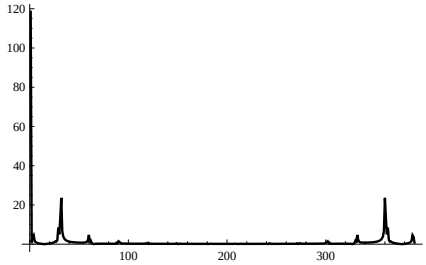

Fig.5e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 32$.

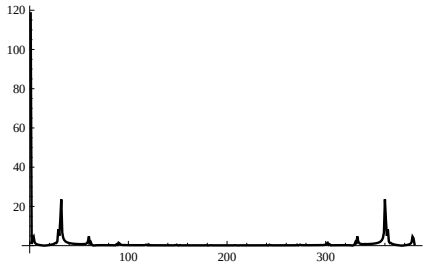

Fig.11f Spectrum of $\theta(t)$ in $0 \le t \le 2\pi \times 32$.

系の規定値は同様とし、初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.1$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 0.1$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$ ($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.12a-12d** に示す.

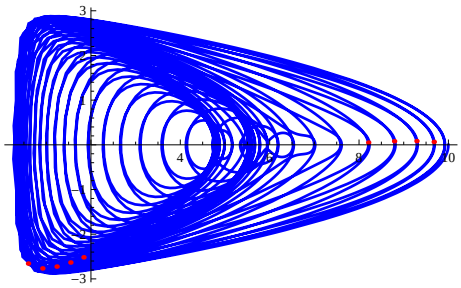

Fig.12a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 32$.

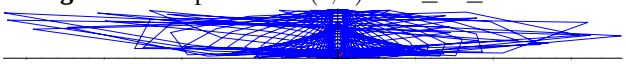

Fig.12b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 32$.The figure is rotated by 90°.

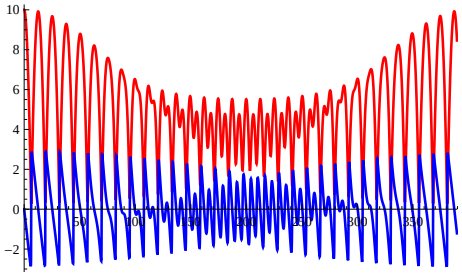

Fig.12c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

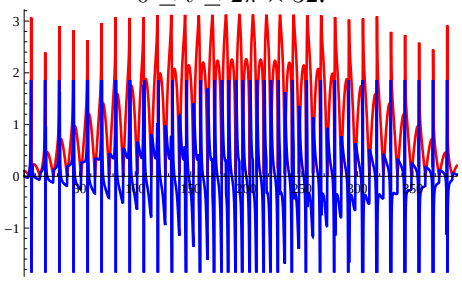

Fig.12d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

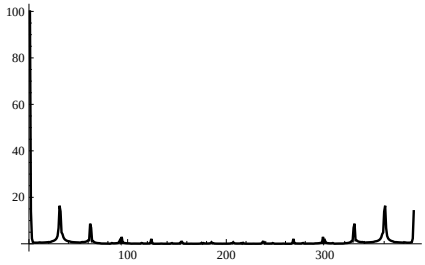

Fig.12e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 32$.

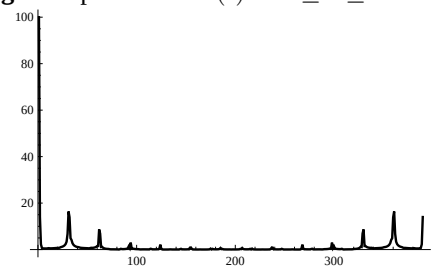

Fig.12f Spectrum of $\theta(t)$ in $0 \le t \le 2\pi \times 32$.

系の規定値は同様とし, 微分方程式系 (2.6), (2.7) 式の初期値を $r(0) = 10$, $\dot{r}(0) = 0$, $\theta(0) = 0.5$, $\dot{\theta}(0) = 0$, $\dot{\varphi}(0) = 1$ と設定して，(2.6), (2.7) 式を時間 $0 \le t \le t_e$

($t_e = 2\pi \times 62$) で数値積分した結果を **Fig.13a-13d** に示す.

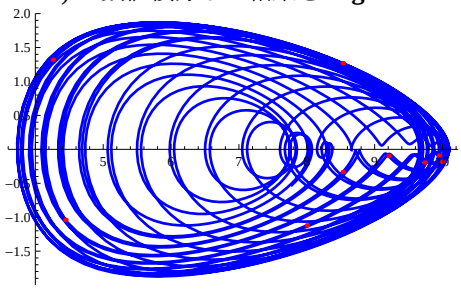

Fig.13a Phase portrait of $(r, \dot{r})$ in $0 \le t \le 2\pi \times 32$.

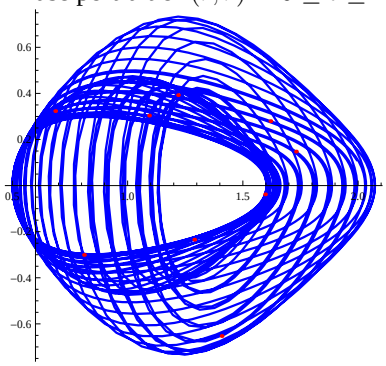

Fig.13b Phase portrait of $(\theta, \dot{\theta})$ in $0 \le t \le 2\pi \times 32$.

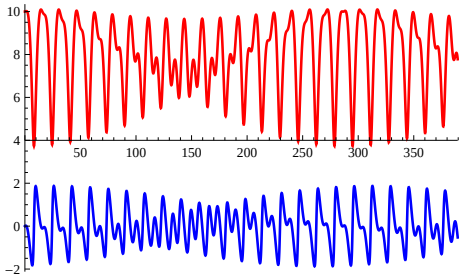

Fig.13c Time sequence of $r(t)$ and $\dot{r}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

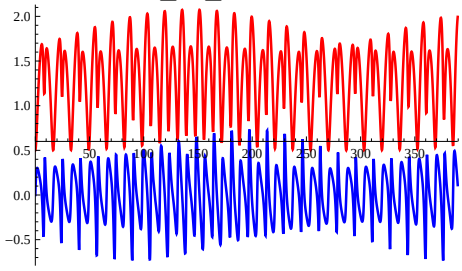

Fig.13d Time sequence of $\theta(t)$ and $\dot{\theta}(t)$ versus t in $0 \le t \le 2\pi \times 32$.

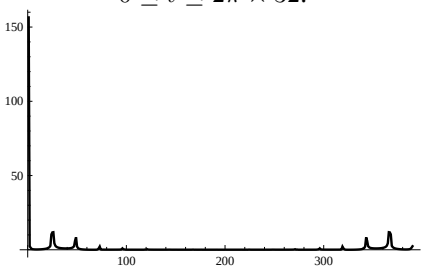

Fig.13e Spectrum of $r(t)$ in $0 \le t \le 2\pi \times 32$.

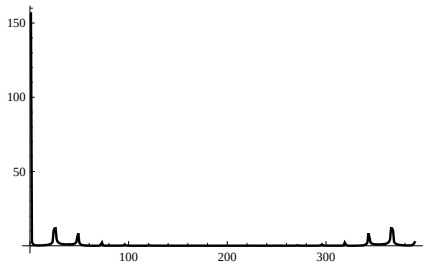

Fig.13f Spectrum of $\theta(t)$ in $0 \le t \le 2\pi \times 32$.

5 おわりに

本報告では球面振子を伴う揺動 Atwood 機械の運動方程式を導き，物体の位置 (r, θ, φ) に対する連立非線形 2 階常微分方程式の力学的特性を論じ，φ 方向の角運動量保存則を用いて運動方程式を書き換え、2 つの修正した運動方程式を数値解析した．球面振子の場合と同様、φ 方向の角運動量の値が non-zero の場合、θ 方向の振動に顕著に影響する。4 節の数値計算例では規則的振動が見られるが、5 節の $\mu > 1$ の場合に多彩な振動現象が起こり，振動の複雑化やカオス化が顕著である。

本研究遂行にあたり，本校の校長リーダーシップ経費による支援を受けたことをここに記して，柳下福蔵校長に厚くお礼申し上げます.

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